APPROXIMATE SYMMETRY AND FORMAL LINEARIZATION

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The methods of classical group analysis make it possible to identify those equations of mathematical physics which are notable for their symmetry properties. Unfortunately, any small perturbation to the equation destroys the symmetry and hence the group, which decreases the practical value of these "special" equations and the group-theoretic methods. Therefore, it is necessary to develop methods of group analysis which are stable against small perturbations of the differential equations. The importance of this problem has been noted on several occasions by L. V. Ovsyannikov. In the preface to his first book on group analysis [1], he stated that "The study of group properties in 'total,' the elucidation of approximate group properties, and other questions, still await solution." In 1974 he noted again [2] that "a general theory of approximate group analysis has yet to be developed."

Recently [3, 4], an analytical theory of approximate symmetries of differential equations with a small parameter was developed. Determining equations for the approximate symmetries were derived and approximate symmetries were constructed for several classes of equations. If the equation

$$F_0 + \varepsilon F_1 \approx 0 \tag{0.1}$$

with small parameter ε approximately admits [to order $o(\varepsilon^p)$] the infinitesimal operator

$$X = X_0 + \varepsilon X_1 + \ldots + \varepsilon^p X_p,\ p \geqslant 1, \tag{0.2}$$

as a symmetry operator, then the unperturbed equation

$$F_0 = 0 \tag{0.3}$$

admits (not approximately, but exactly) the operator X_0. In general, not every operator X_0 admitted as a symmetry operator by (0.3) is inherited by the perturbed equation (0.1), i.e., an approximate symmetry corresponding to the operator (0.2). The complete inheritance by the perturbed equation of the entire symmetry group of the unperturbed equation in the form of an approximate symmetry group can occur in exceptional cases. As shown in [3], it occurs for evolution equations of the type

$$u_t = h(u)u_x + \varepsilon H, \tag{0.4}$$

where $h(u)$ is an arbitrary function; $H = H(t, x, u, u_x, \ldots)$ is an arbitrary element of the space $\mathcal{A}[x, u]$ of differential functions. In this case the order p of the inheritance can be chosen arbitrarily. Hence, one can introduce into consideration a new object: formal symmetry and the intimately connected formal Becklund transformation. The present paper is devoted to the study of these formal symmetries and transformations.

When we go from approximate symmetries to formal symmetries we can remove the condition on the smallness of the parameter ε and consider it as a "graduating" element. The formal symmetries of (0.4) can be represented as a formal power series of the form

$$f = \sum_{i=0}^{\infty} \varepsilon^i f^i,\quad f^i \in \mathcal{A} \tag{0.5}$$

and are constructed recursively with the help of the equation

$$X(u_t - h(u)u_x - \varepsilon H)|_{(0.4)} = 0,\quad X = f\frac{\partial}{\partial u} + \ldots$$

When the parameter ε is small, any finite sum of the series (0.5), which defines a formal symmetry, also defines an approximate symmetry. A special type of formal symmetry is one

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satisfying the cutoff condition of the formal series (0.5). When this condition is fulfilled we obtain the well-known Lie–Becklund symmetries. We note that this approach gives an essentially new method of constructing Lie–Becklund symmetries which differs from the usual method [5] in that the process of constructing the coordinates of the canonical Lie–Becklund operator now goes from the lower derivative terms to the higher, rather than from the higher to the lower. Our approach also explains how it happens that the Lie–Becklund groups of the Burgers and Korteweg–de Vries equations taken separately are not symmetry groups (in the framework of the Lie–Becklund group theory) of the combined Burgers–Korteweg–de Vries equation: it turns out that in the Burgers–Korteweg–de Vries equation these groups become formal symmetries which do not satisfy the series cutoff condition.

Formal (approximate) Becklund transformations are also given by recursively constructed formal power series (i.e., by their finite sums) in ε with coefficients from $\mathcal{A}$. With the help of the formal Becklund transformations it is possible to linearize any evolution equation reducible to the form (0.4). For example, the Korteweg–de Vries equation $u_t = uu_x + u_{xxx}$ is linearized in this way. Earlier in [6] this equation was linearized by means of a formal series and its convergence to the equation $u_t = u_{xxx}$ was discussed. The convergence of this series (in a certain sense) was demonstrated in [8] using the theory of nonlinear Lie group representations [7]. In contrast to the linearization of [6, 8], in our approach the formal series are constructed with explicitly "calculated" coefficients; our approach also brings out new group properties of the Korteweg–de Vries equation.

The following notation is used: t and x are the independent variables; u is a differential variable with successive derivatives (with respect to x) $u_{\alpha+1} = D(u_\alpha)$, $\alpha = 0, 1, 2, \ldots$, $u_0 = u$, where $D=\partial/\partial x+\sum\limits_{\alpha\geqslant 0} u_{\alpha+1}\partial/\partial u_\alpha$; $\mathcal{A}[x, u]$ is the space of differential functions, i.e., analytic functions of an arbitrary finite number of variables $t, x, u, u_1, \ldots$; $f_t = \partial f/\partial t$; $f_x = \partial f/\partial x$; $f_\alpha = \partial f/\partial u_\alpha$; $f_* = \sum\limits_{\alpha\geqslant 0} f_\alpha D^\alpha$.

1. FORMAL SYMMETRY OF THE EQUATION $u_t = h(u)u_x + \varepsilon H$

THEOREM 1.1. All of the symmetries of the equation

$$u_t = h(u)u_1 \tag{1.1}$$

are inherited by the equation

$$u_t = h(u)u_1 + \varepsilon H,\ H \in \mathcal{A} \tag{1.2}$$

and can be expressed in the form of formal symmetries

$$f = \sum_{i\geqslant 0} \varepsilon^i f^i, \quad f^i \in \mathcal{A}. \tag{1.3}$$

Namely, any canonical Lie–Becklund operator $X_0 = f^0\partial/\partial u + \ldots$, admitted by (1.1) corresponds to the operator $X = f\partial/\partial u + \ldots$ with coordinates (1.3), which is admitted by (1.2) as a symmetry operator.

Proof. The determining equation $D_t(f) - h(u)D_x(f) - h'(u)u_1 f = \varepsilon H_* f$ for the infinitesimal operator $X = f\partial/\partial u + \ldots$ admitted by (1.2) takes the following form, after decomposition in powers of ε:

$$f_t^0 - h(u) f_x^0 + \sum_{\alpha\geqslant 1} [D^\alpha(hu_1) - hu_{\alpha+1}] f_\alpha^0 - h'(u) u_1 f^0 = 0; \tag{1.4}$$

$$\begin{gathered} f_t^i - h(u) f_x^i + \sum_{\alpha\geqslant 1} [D^\alpha(hu_1) - hu_{\alpha+1}] f_\alpha^i - \\ - h'(u) u_1 f^i = \sum_{\alpha\geqslant 0} [D^\alpha(f^{i-1}) H_\alpha - f_\alpha^{i-1} D^\alpha(H)], \quad i = 1, 2, \ldots \end{gathered} \tag{1.5}$$

Equation (1.4) for f^0 is the determining equation for the exact group admitted by (1.1). Let f^0 be an arbitrary solution of (1.4) and suppose f^0 is a differential function of order $k_0 \geq 0$, while H is a differential function of order $n \geq 1$, i.e.,

$$f^0 = f^0(t, x, u, \ldots, u_{k_0}), \quad H = H(t, x, u, \ldots, u_n).$$

We write the solution f^1 of (1.5) in the form of a differential function of order $k_1 = n + k_0 - 1$. Then (1.5) will be a linear first-order partial differential equation for the function f^1 of the $k_1 + 3$ arguments $t, x, u, u_1, \ldots, u_{k_1}$, and hence is solvable. Substitution of any solution $f^1(t, x, u, u_1, \ldots, u_{k_1})$ into the right-hand side of (1.5) with $i = 2$ shows that f^2 can be written in the form of a differential function of order $k_2 = n + k_1 - 1$ and the corresponding equation for f_2 is solvable. The higher-order coefficients f^i ($i \geq 3$) of the series (1.3) are found recursively from (1.5). Hence the theorem is proven.

2. FORMAL BECKLUND TRANSFORMATIONS

THEOREM 2.1. Equation (1.2), $u_t = h(u)u_1 + \varepsilon H$, with arbitrary function $H \in \mathscr{A}$ is related to

$$v_t = h(v)v_1 \tag{2.1}$$

by the formal Becklund transformation

$$v = u + \sum_{i\geqslant 1} \varepsilon^i \Phi^i, \quad \Phi^i \in \mathscr{A}. \tag{2.2}$$

Proof. Substitution of (2.2) into (2.1) gives

$$u_t + \sum_{i\geqslant 1} \varepsilon^i D_t(\Phi^i) = h\Big(u + \sum_{i\geqslant 1} \varepsilon^i \Phi^i\Big)\Big[u_1 + \sum_{i\geqslant 1} \varepsilon^i D(\Phi^i)\Big].$$

Hence, using the identity (see [3, Eq. (2.8)])

$$h\Big(u + \sum_{i\geqslant 1} \varepsilon^i \Phi^i\Big) = h(u) + \sum_{j\geqslant 1} \varepsilon^j \sum_{k=1}^{j} \frac{1}{k!} h^{(k)}(u) \sum_{i_1+\ldots+i_k=j} \Phi^{i_1}\ldots\Phi^{i_k}$$

and (1.2), and decomposing with respect to powers of ε, we obtain

$$\Phi_t^1 - h(u)\Phi_x^1 + \sum_{\alpha\geqslant 1} [D^\alpha(hu_1) - hu_{\alpha+1}]\Phi_\alpha^1 - h'(u)u_1\Phi^1 = -H; \tag{2.3}$$

$$\begin{aligned}&\Phi_t^i - h(u)\Phi_x^i + \sum_{\alpha\geqslant 1} [D^\alpha(hu_1) - hu_{\alpha+1}]\Phi_\alpha^i - h'(u)u_1\Phi^i = \\ &= -\sum_{\alpha\geqslant 0} \Phi_\alpha^{i-1} D^\alpha(H) + u_1 \sum_{k=2}^{i} h^{(k)}(u) \sum_{i_1+\ldots+i_k=i} \Phi^{i_1}\ldots\Phi^{i_k} + \\ &+ \sum_{j+l=i} D(\Phi^j)\Big(\sum_{k=1}^{l} \frac{1}{k!} h^{(k)}(u) \sum_{i_1+\ldots+i_k=l} \Phi^{i_1}\ldots\Phi^{i_k}\Big), \quad i\geqslant 2,\end{aligned} \tag{2.4}$$

where the indices $i_1, \ldots, i_k, j, \ell$ takes the values 1, 2, ... Let H be a differential function of order $n \geq 1$, i.e., $H = H(t, x, u, \ldots, u_n)$. We will write the solution Φ^1 of (2.3) as a differential function of order n. Then (2.3) is a linear first-order partial differential equation for the function Φ^1 of the $n + 3$ arguments $t, x, u, u_1, \ldots, u_n$ and therefore is solvable. Substitution of any solution $\Phi^1(t, x, u, u_1, \ldots, u_n)$ into the right-hand side of (2.4) with $i = 2$ shows that Φ^2 can be written as a differential function of order 2n, and the corresponding equation for Φ^2 is solvable. The higher-order coefficients Φ^i of the series (2.2) are determined recursively from (2.4). The theorem is proven.

The following facts are correct for the transfer equation (2.1): the change of variable $\tilde{v} = h(v)$ transforms (2.1) to the "standard" form

$$\tilde{v}_t = \tilde{v}\tilde{v}_1, \tag{2.1'}$$

and (2.1') is linearized using the hodograph transformation $y = \tilde{v}$, $w = t\tilde{v} + x$:

$$w_t = 0. \tag{2.5}$$

Therefore from Theorem 2.1 we have the following important consequence:

Consequence. Equation (2.1) is reduced to the linear equation (2.5) for the function $w = w(t, y)$ by the formal Becklund transformation

$$y = h\Big(u + \sum_{i\geqslant 1} \varepsilon^i\Phi^i\Big), \quad w = x + th\Big(u + \sum_{i\geqslant 1} \varepsilon^i\Phi^i\Big), \tag{2.6}$$

where

$$h\Big(u + \sum_{i\geqslant 1} \varepsilon^i\Phi^i\Big) \equiv h(u) + \varepsilon h'(u)\Phi^1 + \varepsilon^2\Big[h'(u)\Phi^2 + \frac{1}{2}h''(u)(\Phi^1)^2\Big] + \ldots,$$

and the coefficients Φ^1, Φ^2, ... are found recursively from the system (2.3), (2.4).

Note 2.1. The Becklund transformation (2.2) makes it possible to construct the formal symmetries of Eq. (1.2) from the symmetries of (2.1) [and hence from the symmetries of the linear equation (2.5)] without using Theorem 1.1, but using the conversion formula [5]

$$f_u = \Big[1 + \sum_{i\geqslant 1} \varepsilon^i\Phi^i_*\Big]^{-1} f_v \tag{2.7}$$

[f_u and f_v are symmetries (exact or formal) of Eqs. (1.2) and (2.1), respectively].

3. FORMAL RECURRENCE

In the theory of Lie–Becklund groups one introduces recurrence operators, which make it possible to construct the solutions of the determining equations without solving the equations themselves. Similarly, we use formal recurrence operators to construct the formal symmetries. For (1.2), $u_t = h(u)u_1 + \varepsilon H$, $H \in \mathcal{A}$, the formal recurrence operator

$$\tilde{L} = \tilde{\alpha}D + \tilde{\beta} + \tilde{\gamma}D^{-1} + \ldots, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \ldots \in \mathcal{A} \tag{3.1}$$

can be obtained using the formal Becklund transformation (2.2) of the recurrence operator of the unperturbed equation (2.1):

$$L = \frac{\alpha}{h'(v)} D_x \frac{1}{v_1} + \beta + \gamma v_1 D_x^{-1} h'(v) + \ldots \tag{3.2}$$

Here $\alpha, \beta, \gamma, \ldots \in \mathcal{A}[x, v]$ are arbitrary functions of v, $x + th(v)$, $t + 1/(h'(v)v_1)$, ... The conversion formula, relating the recurrence operators of (1.2) and (2.1), takes the form

$$\tilde{L} = \Big(1 + \sum_{i\geqslant 1} \varepsilon^i\Phi^i_*\Big)^{-1} L \Big(1 + \sum_{i\geqslant 1} \varepsilon^i\Phi^i_*\Big). \tag{3.3}$$

Note 3.1. Any finite sum of the power series (3.3) in ε gives an approximate recurrence operator. When the cutoff condition of the series (3.3) is satisfied we obtain the usual recurrence operators.

These methods are used below to study the ordinary and modified Korteweg–de Vries equation and the Burgers–Korteweg–de Vries equation.

4. EXACT AND FORMAL SYMMETRIES OF THE KORTEWEG–DE VRIES EQUATION

We construct the symmetries of the Korteweg–de Vries equation

$$u_t = uu_1 + \varepsilon u_3 \tag{4.1}$$

by the method mentioned in Note 2.1. To do this we find the formal Becklund transformations (2.2) connecting (4.1) to the transfer equation

$$v_t = vv_1. \tag{4.2}$$

We will assume that the coefficients Φ^i in (2.2) do not depend on t and x. Then, since $H = u_3$ in this case, the system (2.3) and (2.4) takes the form

$$\sum_{\alpha\geqslant 1} [D^\alpha(uu_1) - uu_{\alpha+1}]\Phi^1_\alpha - u_1\Phi^1 = -u_3; \tag{4.3}$$

$$\sum_{\alpha\geqslant 1}[D^{\alpha}(uu_1)-uu_{\alpha+1}]\,\Phi_{\alpha}^{i}-u_1\Phi^{i}= \\ =-\sum_{\alpha\geqslant 0}\Phi_{\alpha}^{i-1}u_{\alpha+3}+\sum_{j+l=i}D(\Phi^{j})\,\Phi^{l},\quad i\geqslant 2. \tag{4.4}$$

According to the proof of Theorem 2.1, a particular solution of (4.3) can be written as a differential function of the third order. It follows from (4.4) with i = 2 that Φ^2 is a differential function of the sixth order. The other coefficients Φ^i (i > 2) of the formal transformation (2.2) are found in the same way. Hence,

$$\begin{aligned} v=u&+\varepsilon\left(-\frac{1}{2}\frac{u_3}{u_1}+\frac{1}{2}\frac{u_2^2}{u_1^2}\right)+\varepsilon^2\left(\frac{1}{8}\frac{u_6}{u_1^2}-\frac{29}{40}\frac{u_2u_5}{u_1^3}-\frac{37}{40}\frac{u_3u_4}{u_1^3}+\right.\\ &\left.+\frac{12}{5}\frac{u_2^2u_4}{u_1^4}+\frac{21}{8}\frac{u_2u_3^2}{u_1^4}-\frac{11}{2}\frac{u_2^3u_3}{u_1^5}+2\frac{u_2^5}{u_1^6}\right)+\varepsilon^3\left[-\frac{1}{48}\frac{u_9}{u_1^3}+\frac{19}{80}\frac{u_2u_8}{u_1^4}+\right.\\ &+\frac{269}{560}\frac{u_3u_7}{u_1^4}-\frac{863}{560}\frac{u_2^2u_7}{u_1^5}+\frac{439}{560}\frac{u_4u_6}{u_1^4}-\frac{3207}{560}\frac{u_2u_3u_6}{u_1^5}+\frac{2029}{280}\frac{u_2^3u_6}{u_1^6}+\frac{67}{140}\frac{u_5^2}{u_1^4}-\\ &-\frac{943}{112}\frac{u_2u_4u_5}{u_1^5}-\frac{2679}{560}\frac{u_3^2u_5}{u_1^5}+\frac{2949}{80}\frac{u_2^2u_3u_5}{u_1^6}-\frac{1079}{40}\frac{u_2^4u_5}{u_1^7}-\frac{461}{80}\frac{u_3u_4^2}{u_1^5}+\frac{799}{35}\frac{u_2^2u_4^2}{u_1^6}+\\ &+\left(53-\frac{247}{560}\right)\frac{u_2u_3^2u_4}{u_1^6}-\left(158+\frac{19}{40}\right)\frac{u_2^3u_3u_4}{u_1^7}+\frac{801}{10}\frac{u_2^5u_4}{u_1^8}+\frac{679}{112}\frac{u_3^4}{u_1^6}-\\ &\left.-\frac{375}{4}\frac{u_2^2u_3^3}{u_1^7}+\left(241+\frac{3}{8}\right)\frac{u_2^4u_3^2}{u_1^8}-\left(184+\frac{2}{3}\right)\frac{u_2^6u_3}{u_1^9}+42\frac{u_2^8}{u_1^{10}}\right]+\ldots \end{aligned} \tag{4.5}$$

To transform a symmetry f_v of (4.2) into a symmetry f_u of the Korteweg–de Vries equation (4.1) by means of (4.5), we use (2.7). Using (4.5) and

$$(1+\varepsilon A+\varepsilon^2B+\ldots)^{-1}=1-\varepsilon A+\varepsilon^2(A^2-B)+\ldots \tag{4.6}$$

(2.7) can be written in the form

$$\begin{aligned} f_u=\Bigg\{1&+\varepsilon\left[\frac{1}{2u_1}D^3-\frac{u_2}{u_1^2}D^2-\frac{1}{2}\frac{u_3}{u_1^2}D+\frac{u_2^2}{u_1^3}D\right]+\\ &+\varepsilon^2\left[\frac{1}{8u_1^2}D^6-\frac{41}{40}\frac{u_2}{u_1^3}D^5+\left(-\frac{73}{40}\frac{u_3}{u_1^3}+\frac{51}{10}\frac{u_2^2}{u_1^4}\right)D^4+\right.\\ &+\left(-\frac{63}{40}\frac{u_4}{u_1^3}+\frac{27}{2}\frac{u_2u_3}{u_1^4}-17\frac{u_2^3}{u_1^5}\right)D^3+\\ &+\left(-\frac{21}{40}\frac{u_5}{u_1^3}+\frac{36}{5}\frac{u_2u_4}{u_1^4}+\frac{45}{8}\frac{u_3^2}{u_1^4}-\frac{87}{2}\frac{u_2^2u_3}{u_1^5}+35\frac{u_2^4}{u_1^6}\right)D^2+\\ &\left.+\left(\frac{33}{40}\frac{u_2u_5}{u_1^4}+\frac{99}{40}\frac{u_3u_4}{u_1^4}-\frac{99}{10}\frac{u_2^2u_4}{u_1^5}-\frac{33}{2}\frac{u_2u_3^2}{u_1^5}+55\frac{u_2^3u_3}{u_1^6}-33\frac{u_2^5}{u_1^7}\right)D\right]+\ldots\Bigg\}f_v. \end{aligned} \tag{4.7}$$

Application of (4.7) to the point symmetries f_v corresponding to Galilean transformations and scaling transformations for the transfer equation (4.2), converts them into corresponding point symmetries of (4.1), i.e., in these cases the cutoff condition of the series (4.7) is satisfied.

We consider now the point symmetries

$$f_v=\varphi(v)v_1, \tag{4.8}$$

which give the nonlinear form for (4.2) of the principle of linear superposition for (2.5), which states that (2.5) is invariant when an arbitrary function φ(y) is added to w.

If $\varphi(v) = v^2$, then (4.7) gives

$$f_u=u^2u_1+\varepsilon[4u_1u_2+2uu_3+(6/5)u_5]+\ldots \tag{4.9}$$

According to [3], the cutoff condition for the series (4.9) is satisfied and, therefore, the point symmetry $f_v = v^2 v_1$ transforms into the well-known [5, p. 191] Lie–Becklund symmetry of (4.1). Similarly, it can be shown that for the point symmetries $f_v = v^n v_1$ with integer $n > 2$ the formal series f_u satsifies the cutoff condition and gives the well-known Lie–Becklund symmetries of (4.1) (see also [3]).

All of the remaining point symmetries of the transfer equation (4.2) [and also symmetries obtainable from them using the recurrence operators (3.2)] transform into formal symmetries of the Korteweg–de Vries equation. For example, the symmetry (4.8) in the case of a nonpolynomial function $\varphi(v)$ transforms into the formal symmetry

$$f_u = \varphi(u)\,u_1 + \varepsilon\left(\varphi' u_3 + 2\varphi'' u_1 u_2 + \frac{1}{2}\varphi''' u_1^3\right) + \varepsilon^2\left(\frac{3}{5}\varphi'' u_5 + \frac{9}{5}\varphi''' u_1 u_4 + 3\varphi''' u_2 u_3 + \frac{23}{10}\varphi^{IV} u_1^2 u_3 + \frac{31}{10}\varphi^{IV} u_1 u_2^2 + \frac{8}{5}\varphi^{V} u_1^3 u_2 + \frac{1}{8}\varphi^{VI} u_1^5\right) + \ldots$$

5. EXACT AND FORMAL RECURRENCES OF THE KORTEWEG–DE VRIES EQUATION

We transform the recurrence operator L of (3.2) for (4.2) into the formal recurrence operator $\tilde{L}$ for (4.1) using the transformation (4.5). According to (3.3), the operators L and $\tilde{L}$ are connected by the formula

$$\tilde{L} = \Phi_*^{-1} L \Phi_*, \tag{5.1}$$

where

$$\Phi_* = 1 + \varepsilon D\left(-\frac{1}{2u_1}D^2 + \frac{u_2}{2u_1^2}D\right) + \varepsilon^2 D\left(\frac{1}{8u_1^2}D^5 - \frac{19}{40}\frac{u_2}{u_1^3}D^4 - \frac{9}{20}\frac{u_3}{u_1^3}D^3 + \frac{39}{40}\frac{u_2^2}{u_1^4}D^3 - \frac{19}{40}\frac{u_4}{u_1^3}D^2 + \frac{39}{20}\frac{u_2 u_3}{u_1^4}D^2 - \frac{8}{5}\frac{u_2^3}{u_1^5}D^2 - \frac{1}{4}\frac{u_5}{u_1^3}D + \frac{57}{40}\frac{u_2 u_4}{u_1^4}D + \frac{27}{40}\frac{u_3^2}{u_1^4}D - \frac{39}{10}\frac{u_2^2 u_3}{u_1^5}D + 2\frac{u_2^4}{u_1^6}D\right) + \ldots,$$

and Φ_*^{-1} is found with the use of (4.6).

For example, substitution of the operator $L_1 = \beta(v)$ into (5.1) and the use of the relation

$$\beta(v) = \beta(u) + \varepsilon\beta'(u)\left(-\frac{1}{2}\frac{u_3}{u_1} + \frac{1}{2}\frac{u_2^2}{u_1^2}\right) + \varepsilon^2\left[\beta'(u)\left(\frac{1}{8}\frac{u_6}{u_1^2} - \frac{29}{40}\frac{u_2 u_5}{u_1^3} - \frac{37}{40}\frac{u_3 u_4}{u_1^3} + \frac{12}{5}\frac{u_2^2 u_4}{u_1^4} + \frac{21}{8}\frac{u_2 u_3^2}{u_1^4} - \frac{11}{2}\frac{u_2^3 u_3}{u_1^5} + 2\frac{u_2^5}{u_1^6}\right) + \frac{1}{8}\beta''(u)\left(\frac{u_3^2}{u_1^2} - 2\frac{u_2^2 u_3}{u_1^3} + \frac{u_2^4}{u_1^4}\right)\right] + \ldots$$

gives

$$\tilde{L}_1 = \Phi_*^{-1} L_1 \Phi_* = \beta(u) + \varepsilon\left[\frac{3}{2}\beta' D^2 - \frac{1}{2}\frac{\beta' u_2}{u_1}D + \frac{3}{2}\beta'' u_1 D - \frac{\beta' u_3}{2u_1} + \frac{\beta' u_2^2}{2u_1^2} + \frac{1}{2}\beta'' u_2 + \frac{1}{2}\beta''' u_1^2\right] + \varepsilon^2\left[\frac{9}{8}\beta'' D^4 - \frac{3}{10}\frac{\beta' u_3}{u_1^2}D^3 + \frac{2}{5}\frac{\beta' u_2^2}{u_1^3}D^3 + \frac{9}{4}\beta''' u_1 D^3 - \frac{3}{4}\frac{\beta'' u_2}{u_1}D^3 - \frac{3}{5}\frac{\beta' u_4}{u_1^2}D^2 + \frac{14}{5}\frac{\beta' u_2 u_3}{u_1^3}D^2 - \frac{12}{5}\frac{\beta' u_2^3}{u_1^4}D^2 - \frac{39}{20}\frac{\beta'' u_3}{u_1}D^2 + \frac{89}{40}\frac{\beta'' u_2^2}{u_1^2}D^2 + \frac{3}{2}\beta''' u_2 D^2 + \frac{15}{8}\beta^{IV} u_1^2 D^2 - \frac{3}{10}\frac{\beta' u_5}{u_1^2}D + \frac{23}{10}\frac{\beta' u_2 u_4}{u_1^3}D + \frac{17}{10}\frac{\beta' u_3^2}{u_1^3}D - \frac{48}{5}\frac{\beta' u_2^2 u_3}{u_1^4}D + 6\frac{\beta' u_2^4}{u_1^5}D - \frac{69}{40}\frac{\beta'' u_4}{u_1}D + \tag{5.2}$$

$$+\frac{61}{10}\frac{\beta''u_2u_3}{u_1^2}D-\frac{177}{40}\frac{\beta''u_2^3}{u_1^3}D-\frac{3}{10}\beta'''u_3D+\frac{23}{20}\frac{\beta'''u_2^2}{u_1}D+\frac{3}{2}\beta^{IV}u_1u_2D+ \tag{5.2}$$

$$+\frac{3}{4}\beta^{V}u_1^3D+\frac{3}{10}\frac{\beta'u_2u_5}{u_1^3}+\frac{9}{10}\frac{\beta'u_3u_4}{u_1^3}-\frac{27}{10}\frac{\beta'u_2^2u_4}{u_1^4}-\frac{9}{2}\frac{\beta'u_2u_3^2}{u_1^4}+12\frac{\beta'u_2^3u_3}{u_1^5}-$$

$$-6\frac{\beta'u_2^5}{u_1^6}-\frac{21}{40}\frac{\beta''u_5}{u_1}+\frac{99}{40}\frac{\beta''u_2u_4}{u_1^2}+\frac{39}{20}\frac{\beta''u_3^2}{u_1^2}-\frac{1041}{8}\frac{\beta''u_2^2u_3}{u_1^3}+\frac{177}{40}\frac{\beta''u_2^4}{u_1^4}-$$

$$-\frac{9}{20}\beta'''u_4+\frac{9}{5}\frac{\beta'''u_2u_3}{u_1}-\frac{23}{20}\frac{\beta'''u_2^3}{u_1^2}+\frac{17}{40}\beta^{IV}u_1u_3+\frac{29}{40}\beta^{IV}u_2^2+$$

$$\left.+\frac{17}{20}\beta^{V}u_1^2u_2+\frac{1}{8}\beta^{VI}u_1^4\right]+\ldots$$

Similarly, for the operator $L_2 = v_1D^{-1}$ we have

$$\tilde{L}_2=\Phi_*^{-1}L_2\Phi_*=u_1D^{-1}+\varepsilon\left[\frac{u_2}{u_1}D+\frac{u_3}{u_1}-\frac{u_2^2}{u_1^2}\right]+\varepsilon^2\left[\frac{3}{5}\frac{u_3}{u_1^2}D^3-\right.$$

$$-\frac{4}{5}\frac{u_2^2}{u_1^3}D^3+\frac{6}{5}\frac{u_4}{u_1^2}D^2-\frac{28}{5}\frac{u_2u_3}{u_1^3}D^2+\frac{24}{5}\frac{u_2^3}{u_1^4}D^2+\frac{3}{5}\frac{u_5}{u_1^2}D- \tag{5.3}$$

$$-\frac{23}{5}\frac{u_2u_4}{u_1^3}D-\frac{17}{5}\frac{u_3^2}{u_1^3}D+\frac{96}{5}\frac{u_2^2u_3}{u_1^4}D-12\frac{u_2^4}{u_1^5}D-\frac{3}{5}\frac{u_2u_5}{u_1^3}-$$

$$\left.-\frac{9}{5}\frac{u_3u_4}{u_1^3}+\frac{27}{5}\frac{u_2^2u_4}{u_1^4}+9\frac{u_2u_3^2}{u_1^4}-24\frac{u_2^3u_3}{u_1^5}+12\frac{u_2^5}{u_1^6}\right]+\ldots$$

It follows from (5.2) and (5.3) that for $\beta(u) = u$ the formal recurrence $\tilde{L} = \tilde{L}_1 + 2\tilde{L}_2$ is given by the cutoff series and coincides with the well-known exact recurrence $\tilde{L} = u + 2u_1D^{-1} + 3\varepsilon D^2$ of the Korteweg–de Vries equation (4.1).

6. ON THE CUTOFF OF THE FORMAL SYMMETRY SERIES OF THE MODIFIED KORTEWEG–DE VRIES EQUATION

For the equation

$$u_t = h(u)u_1 + \varepsilon u_3 \tag{6.1}$$

we consider the formal symmetries (1.3) with coefficients f^i which are independent of t and x. It follows from Theorem 1.1 that if f^0 is a symmetry of (1.1), then the coefficients f^1, f^2, ... are given by the determining equations

$$\sum_{\alpha\geqslant 1}[D^{\alpha}(hu_1)-hu_{\alpha+1}]f^i_{\alpha}-h'(u)u_1f^i= \\ =\sum_{\alpha\geqslant 0}[2D(f^{i-1}_{\alpha})u_{\alpha+2}+D^2(f^{i-1}_{\alpha})u_{\alpha+1}],\quad i\geqslant 1. \tag{6.2}$$

It can be shown from the solutions of (6.2) that in the case $h'''(u) = 0$ the cutoff condition of the series (1.3) is satisfied if $f^0 = \varphi(u)u_1$, where $\varphi(u)$ is a polynomial function. This is consistent with the well-known fact that the Lie–Becklund group exists for the ordinary and modified Korteweg–de Vries equations (see [5, p. 215], for example). Apparently the cutoff condition for the formal series (1.3) is satisfied only when h(u) is of the form $h = C_1u + C_2u^2$; this is suggested by the structure of the first few terms of this series, obtained by solving (6.2):

$$f^1=\psi_1u_3+2\psi_1'u_1u_2+\frac{1}{2}\psi_1''u_1^3,$$

$$f^2=\frac{3}{5}\psi_2u_5+\frac{9}{5}\psi_2'u_1u_4+3\psi_2'u_2u_3+\left(\frac{23}{10}\psi_2''+\frac{2}{5}\psi_3h''-\frac{1}{10}\psi_2\frac{h'''}{h'}\right)u_1^2u_3+$$

$$+\left(\frac{31}{10}\psi_2'' + \frac{4}{5}\psi_3 h'' - \frac{1}{5}\psi_2\frac{h'''}{h'}\right)u_1u_2^2 + \left(\frac{8}{5}\psi_2''' + \frac{4}{5}\psi_3' h'' + \frac{3}{5}\psi_3 h''' -\right.$$

$$\left. - \frac{1}{5}\psi_2\frac{h^{\mathrm{IV}}}{h'} + \frac{1}{5}\psi_2\frac{h''h'''}{(h')^2}\right)u_1^3u_2 + \frac{1}{8}\left(\frac{8}{5}\psi_2^{\mathrm{IV}} + \frac{4}{5}\psi_3'' h'' + \frac{7}{5}\psi_3 h''' + \frac{2}{3}\psi_3 h^{\mathrm{IV}} +\right.$$

$$\left. + \frac{1}{5}\psi_3\frac{h''h'''}{h'} + \frac{2}{5}\psi_2\frac{h''h^{\mathrm{IV}}}{(h')^2} - \frac{1}{5}\psi_2\frac{h^{\mathrm{V}}}{h'} + \frac{1}{5}\psi_2\frac{(h''')^2}{(h')^2} - \frac{2}{5}\psi_2\frac{(h'')^2h'''}{(h')^3}\right)u_1^5,$$

$$f^3 = \frac{9}{35}\psi_3 u_7 + \frac{36}{35}\psi_3' u_1u_6 + \left(\frac{12}{5}\psi_3' - \frac{3}{40}\psi_2\frac{h'''}{(h')^2}\right)u_2u_5 + \frac{1}{7}\left(\frac{129}{10}\psi_3'' -\right.$$

$$\left. - \frac{3}{5}\psi_3\frac{h'''}{h'} + 3\psi_4 h'' - \frac{3}{10}\psi_2\frac{h^{\mathrm{IV}}}{(h')^2} + \frac{3}{8}\psi_2\frac{h''h'''}{(h')^2}\right)u_1^2u_5 + \left(\frac{18}{5}\psi_3' - \frac{9}{40}\psi_2\frac{h'''}{(h')^2}\right)u_3u_4 + \ldots$$

Here $\psi_1(u) = \varphi'(u)/h'(u)$, $\psi_k(u) = \psi_{k-1}'(u)/h'(u)$, $k = 2, 3, \ldots$

7. FORMAL SYMMETRIES OF THE BURGERS–KORTEWEG–DE VRIES EQUATION

For the Burgers–Korteweg–de Vries equation

$$u_t = uu_1 + \varepsilon(au_3 + bu_2),\ a,\ b = \mathrm{const} \tag{7.1}$$

we find the transformation reducing it to the transfer equation (4.2) and we construct the formal symmetry associated with the point symmetry $f_v = \varphi(v)v_1$ of (4.2).

In this case we have $h = u$, $H = au_3 + bu_2$ and the solution of the system (2.3), (2.4) under the condition that the Φ^i are independent of t and x gives

$$v = u + \varepsilon\left(-\frac{a}{2}\frac{u_3}{u_1} + \frac{a}{2}\frac{u_2^2}{u_1^2} - b\frac{u_2}{u_1}\right) + \varepsilon^2\left(\frac{a^2}{8}\frac{u_6}{u_1^2} - \frac{29}{40}a^2\frac{u_2u_5}{u_1^3} -\right.$$

$$- \frac{37}{40}a^2\frac{u_3u_4}{u_1^3} + \frac{12}{5}a^2\frac{u_2^2u_4}{u_1^4} + \frac{21}{8}a^2\frac{u_2u_3^2}{u_1^4} - \frac{11}{2}a^2\frac{u_2^3u_3}{u_1^5} + 2a^2\frac{u_2^5}{u_1^6} +$$

$$\left. + \frac{1}{2}ab\frac{u_5}{u_1^2} - \frac{9}{4}ab\frac{u_2u_4}{u_1^3} - \frac{5}{4}ab\frac{u_3^2}{u_1^3} + \frac{57}{10}ab\frac{u_2^2u_3}{u_1^4} - \frac{13}{5}ab\frac{u_2^4}{u_1^5} + \frac{1}{2}b^2\frac{u_4}{u_1^2} - \frac{5}{3}b^2\frac{u_2u_3}{u_1^3} + b^2\frac{u_2^3}{u_1^4}\right) + \ldots$$

In this case (2.7) takes the form

$$f_u = \left\{1 + \varepsilon\left[\frac{a}{2u_1}D^3 + \left(-a\frac{u_2}{u_1^2} + \frac{b}{u_1}\right)D^2 + \left(-\frac{a}{2}\frac{u_3}{u_1^2} + a\frac{u_2^2}{u_1^3} -\right.\right.\right.$$

$$\left.\left. - b\frac{u_2}{u_1^2}\right)D\right] + \varepsilon^2\left[\frac{a^2}{8u_1^2}D^6 + \left(-\frac{41}{40}a^2\frac{u_2}{u_1^3} + \frac{1}{2}ab\frac{1}{u_1^2}\right)D^5 + \left(-\frac{73}{40}a^2\frac{u_3}{u_1^3} +\right.\right.$$

$$\left. + \frac{51}{10}a^2\frac{u_2^2}{u_1^4} - \frac{13}{4}ab\frac{u_2}{u_1^3} + \frac{1}{2}b^2\frac{1}{u_1^2}\right)D^4 + \left(-\frac{63}{40}a^2\frac{u_4}{u_1^3} + \frac{27}{2}a^2\frac{u_2u_3}{u_1^4} -\right.$$

$$\left. - 17a^2\frac{u_2^3}{u_1^5} - 4ab\frac{u_3}{u_1^3} + \frac{59}{5}ab\frac{u_2^2}{u_1^4} - \frac{7}{3}b^2\frac{u_2}{u_1^3}\right)D^3 + \left(-\frac{21}{40}a^2\frac{u_5}{u_1^3} +\right.$$

$$+ \frac{36}{5}a^2\frac{u_2u_4}{u_1^4} + \frac{45}{8}a^2\frac{u_3^2}{u_1^4} - \frac{87}{2}a^2\frac{u_2^2u_3}{u_1^5} + 35a^2\frac{u_2^4}{u_1^6} - \frac{7}{4}ab\frac{u_4}{u_1^3} + \frac{181}{10}ab\frac{u_2u_3}{u_1^4} - \tag{7.2}$$

$$\left. - \frac{123}{5}ab\frac{u_2^3}{u_1^5} - \frac{4}{3}b^2\frac{u_3}{u_1^3} + 5b^2\frac{u_2^2}{u_1^4}\right)D^2 + \left(\frac{33}{40}a^2\frac{u_2u_5}{u_1^4} + \frac{99}{40}a^2\frac{u_3u_4}{u_1^4} -\right.$$

$$- \frac{99}{10}a^2\frac{u_2^2u_4}{u_1^5} - \frac{33}{2}a^2\frac{u_2u_3^2}{u_1^5} + 55a^2\frac{u_2^2u_3}{u_1^6} - 33a^2\frac{u_2^5}{u_1^7} + \frac{11}{4}ab\frac{u_2u_4}{u_1^4} + \frac{11}{4}ab\frac{u_3^2}{u_1^4} -$$

$$\left.\left.\left. - \frac{121}{5}ab\frac{u_2^2u_3}{u_1^5} + 22ab\frac{u_2^4}{u_1^6} + 2b^2\frac{u_2u_3}{u_1^4} - 4b^2\frac{u_2^3}{u_1^5}\right)D\right] + \ldots\right\}f_v.$$

For $f_v = \varphi(v)v_1$ we have, according to (7.2),

$$f_u = \varphi(u)u_1 + \varepsilon\left(a\varphi' u_3 + 2a\varphi'' u_1u_2 + \frac{1}{2}a\varphi''' u_1^3 + b\varphi' u_3 + b\varphi'' u_1^2\right) +$$

$$+\varepsilon^2\left(\frac{3}{5}a^2\varphi''u_5+\frac{5}{4}ab\varphi''u_4+\frac{1}{10}ab\varphi''\frac{u_2u_3}{u_1}-\frac{1}{20}ab\varphi''\frac{u_2^3}{u_1^2}+\frac{2}{3}b^2\varphi''u_3+\right.$$
$$+\frac{9}{5}a^2\varphi'''u_1u_4+3a^2\varphi'''u_2u_3+\frac{7}{2}ab\varphi'''u_1u_3+\frac{23}{10}ab\varphi'''u_2^2+$$
$$+\frac{5}{3}b^2\varphi'''u_1u_2+\frac{23}{10}a^2\varphi^{IV}u_1^2u_3+\frac{31}{10}a^2\varphi^{IV}u_1u_2^2+\frac{15}{4}ab\varphi^{IV}u_1^2u_2+$$
$$\left.+\frac{1}{2}b^2\varphi^{IV}u_1^3+\frac{8}{5}a^2\varphi^{V}u_1^3u_2+\frac{1}{2}ab\varphi^{V}u_1^4+\frac{1}{8}a^2\varphi^{VI}u_1^5\right)+\ldots$$

From this expression and from analysis of the transfer formula (2.7) we see how the Lie–Becklund symmetry for the Korteweg–de Vries and Burgers equations transforms into a formal symmetry (7.2) for the equation (7.1) that does not satisfy the cutoff condition.

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QUANTITATIVE CHARACTERISTICS OF THE MAIN CONCEPTS OF LINEAR CONTROL THEORY

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In classical linear control theory there is detailed study of the possibility of selecting a control u(t) which would make it possible to obtain some optimum behavior of trajectory x(t) described by the system

$$\frac{d}{dt}x(t)=Ax(t)+Bu(t). \tag{1}$$

Normally it is assumed to be possible to obtain information about the behavior of this trajectory only from the vector of observation z(t) = Cx(t). We limit ourselves to considering a particular, but important in many typical cases, independent of time t, matrix A, B, C. There is extensive use (see, e.g., [1-3]) of the concept of controllability for pair A, B and the dual concept to each other of stabilizability for A, B and detectability for pair A, C. (If A, B is controllable or stabilizable, then A*, B* is observable and detectable, and conversely).

We introduce criteria (necessary and sufficient) for controllability and stabilizability. Pair A, B is controllable if the linear shell of columns for the composite matrix

$$(B \vdots AB \vdots A^2B \vdots \ldots \vdots A^{N-1}B) \tag{2}$$

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